

# Rotating Stars in Relativity

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**Abstract.** Rotating relativistic stars are receiving significant attention in recent years, because of the information they can yield about the equation of state of matter at extremely high densities and because they are one of the more possible sources of detectable gravitational waves. We review the latest theoretical and numerical methods for modeling rotating relativistic stars, including stars with a strong magnetic field and hot proto-neutron stars. We also review nonaxisymmetric oscillations and instabilities in rotating stars and summarize the latest developments regarding the gravitational wave-driven (CFS) instability in both polar and axial quasi-normal modes.

## 1 Introduction

Rotating relativistic stars are of fundamental interest in physics. Their bulk properties restrict the proposed possible equations of state for densities larger than nuclear density. Their oscillations can become unstable, producing gravitational waves that could be detectable, providing thereby a new way of probing the interior of neutron stars.

Recent research has considerably advanced our understanding of these objects. There now exist several independent numerical codes for obtaining accurate models of rotating neutron stars in full general relativity. Three of these codes have been shown to agree with each other to remarkable accuracy and one code is available as public domain for use by other researchers.

The numerically constructed maximum mass models, for different proposed equations of state, differ by as much as a factor of two in mass, radius and angular velocity, a factor of five in central density and a factor of eight in the moment of inertia. These large uncertainties show that

our understanding of the properties of matter at very high densities is currently rather poor.

Despite the different maximum rotation rates, corresponding to different candidates for the equation of state of neutron-star matter, one can place an absolute upper limit on the rotation of relativistic stars, by imposing causality as the only requirement on the equation of state. It then follows that gravitationally bound stars cannot rotate faster than 0.28 ms.

Although observed magnetic fields in neutron stars have a negligible effect on neutron-star structure, a sufficiently strong magnetic field acts as a centrifugal force on a relativistic star, flattening its shape and increasing the maximum mass and rotation rate for a given equation of state. The magnetic field strength of a stationary configuration has been shown to have an upper limit of  $B \sim 10^{17}$  G.

Rapidly rotating proto-neutron stars are shown to have an extended envelope, due to their high temperature and the presence of trapped neutrinos. If the equation of state is softened, as the neutron star cools, by a large amplitude phase transition, then the nascent neutron star may collapse to a black hole. A surprising result is that a supramassive proto-neutron star, even though it contracts during cooling, evolves to a cold neutron star of smaller angular velocity.

In rotating stars, nonaxisymmetric perturbations have been studied in the Newtonian and post-Newtonian approximations, in the slow-rotation limit and in the Cowling approximation but fully relativistic quasi-normal modes (except for neutral modes) have yet to be obtained. The effect of rotation on the quasi-normal modes of oscillation is to couple polar and axial modes and to shift their frequencies and damping times causing some modes to become unstable.

Nonaxisymmetric instabilities in rotating stars can be driven by the emission of gravitational waves (CFS-instability) or by viscosity. The onset of the CFS-instability has now been computed for fully relativistic, rapidly rotating stars. Relativity has a strong influence on the onset of the instability, allowing it to occur for less rapidly rotating stars than was suggested by Newtonian computations.

Contrary to what was previously thought, nascent neutron stars can be subject to the  $l = 2$  bar mode CFS-instability, emitting strong gravitational waves. The frequency of the waves sweeps downward through the optimal LIGO sensitivity window and first estimates show that it could be detectable out to the distance of 140 Mpc by the advanced LIGO detector.

The viscosity-driven instability is not favored by general relativity but, as a new relativistic computation shows, is absent in rotating neutron stars, unless the equation of state is unexpectedly stiff.

Axial fluid modes in rotating stars ( $r$ -modes) received renewed attention since it was discovered that they are generically unstable to the emission of gravitational waves. The  $r$ -mode instability can slow down a newly-born rapidly rotating neutron star to Crab-like rotation rates. First results show that, the gravitational waves from the spin-down (directly, or as a stochastic background) could be detectable by the advanced LIGO or VIRGO detectors.

The present article aims at presenting a summary of theoretical and numerical methods that are used to describe the equilibrium properties of rotating relativistic stars and their oscillations. It focuses on the most recently available preprints, in order to rapidly communicate new methods and results. At the end of some sections, the reader is pointed to papers that could not be presented in detail here. As new developments in the field occur, updated versions of this article will appear.

## 2 The Equilibrium Structure of Rotating Relativistic Stars

### 2.1 Assumptions

Although a relativistic star has a complicated structure (solid crust, magnetic field, possible superfluid interior etc.), its bulk properties can be computed with reasonable accuracy by making several simplifying assumptions.

The matter is modeled as a perfect fluid because observations of pulsar glitches have shown that the departures from perfect fluid equilibrium due to the solid crust are of order  $10^{-5}$  [1]. The temperature of a cold neutron star is assumed to be 0 K because its thermal energy ( $\ll 1$  MeV  $\sim 10^{10}$  K) is much smaller than the Fermi energy of the interior ( $> 60$  MeV). One can then use a zero-temperature (one-parameter) equation of state (EOS) to describe the

matter:

$$\epsilon = \epsilon(P), \quad (1)$$

where  $\epsilon$  is the energy density and  $P$  is the pressure. At birth, a neutron star is differentially rotating but as the neutron star cools, shear viscosity, resulting from neutrino diffusion, aided by convective and turbulent motions and possibly by the winding-up of magnetic field lines, enforces uniform rotation. At present, it is difficult to accurately compute the timescale in which uniform rotation is enforced, but it is estimated to be of the order of seconds [2].

Within roughly a year after its formation, the neutron star temperature becomes less than  $10^9$  K and its outer core becomes superfluid (see [3] and references therein). Rotation causes the superfluid neutrons to form an array of quantized vortices, with an intervortex spacing of

$$d_n \sim 3.4 \times 10^{-3} \Omega_2^{-1/2} \text{ cm}, \quad (2)$$

where  $\Omega_2$  is the angular velocity of the star in  $10^2 \text{ s}^{-1}$ . On scales much larger than the intervortex spacing, e.g. on the order of 1 cm, the fluid motions can be averaged and the rotation can be considered uniform [4]. The error in computing the metric is of order

$$\left(\frac{1 \text{ cm}}{R}\right)^2 \sim 10^{-11}, \quad (3)$$

where  $R$  is a typical neutron star radius [1].

The above arguments show that the bulk properties of a rotating relativistic star can be modeled accurately by a uniformly rotating, zero-temperature perfect fluid.

### 2.2 Geometry of Space-Time

In relativity, the space-time geometry of a rotating star in equilibrium is described by a stationary and axisymmetric metric of the form

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\phi - \omega dt)^2 + e^{2\alpha} (dr^2 + r^2 d\theta^2), \quad (4)$$

where  $\nu$ ,  $\psi$ ,  $\omega$  and  $\alpha$  are four metric functions which depend on the coordinates  $r$  and  $\theta$  only (unless otherwise noted, we assume  $c = G = 1$ ). The perfect fluid has a stress-energy tensor

$$T^{ab} = (\epsilon + P) u^a u^b + P g^{ab}, \quad (5)$$

a four velocity

$$u^a = \frac{e^{-\nu}}{\sqrt{1 - \omega^2}} (t^a + \Omega \phi^a), \quad (6)$$

and a 3-velocity with respect to a zero angular momentum observer of

$$v = (\Omega - \omega) e^{\psi - \nu}, \quad (7)$$

where  $t^a$  and  $\phi^a$  are the two killing vectors associated with the time and translational symmetries of the space-time,  $g_{ab}$  is the metric tensor and  $\Omega$  is the angular velocity.

Having specified an equation of state for very dense matter, the structure of the star is computed by solving four components of Einstein's gravitational field equations

$$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T), \quad (8)$$

(where  $R_{ab}$  is the Ricci tensor and  $T = T_a^a$ ) and the equation of hydrostationary equilibrium.

## 2.3 Equations of State

The simplest equation of state one can use to model relativistic stars is the relativistic polytropic EOS [5]

$$P = K\rho^\Gamma, \quad (9)$$

$$\epsilon = \rho c^2 + \frac{P}{\Gamma - 1}, \quad (10)$$

where  $\rho$  is the rest mass density,  $K$  is a constant and  $\Gamma$  is the polytropic exponent. Instead of  $\Gamma$ , one often uses the polytropic index  $N$ , defined through

$$\Gamma = 1 + \frac{1}{N}. \quad (11)$$

For this equation of state, the quantity  $c^{(\Gamma-2)/(\Gamma-1)}\sqrt{K^{1/(\Gamma-1)}/G}$  has units of length. In gravitational units ( $c = G = 1$ ), one can thus use  $K^{N/2}$  as a fundamental length scale to define dimensionless quantities. Equilibrium models are then characterized by the polytropic index  $N$  and their properties can be scaled to different values, using an appropriate value for  $K$ . For  $N < 1.0$  ( $N > 1.0$ ) one obtains stiff(soft) models, while for  $N = 0.5 - 1.0$ , one obtains models with bulk properties that are comparable to those of observed neutron stars.

Note that for the above polytropic EOS, the polytropic index  $\Gamma$  coincides with the adiabatic index of a relativistic isentropic fluid

$$\Gamma = \frac{\epsilon + P}{P} \frac{dP}{d\epsilon}. \quad (12)$$

This is not the case for the polytropic equation of state  $P = K\epsilon^\Gamma$ , that has been used by other authors, which satisfies (12) only in the Newtonian limit.

The true equation of state that describes the interior of compact stars is largely unknown. This results from the inability to verify experimentally the different theories that describe the strong interactions between baryons and the many-body theories of dense matter at densities larger than about twice the nuclear density (i.e. at densities larger than about  $5 \times 10^{14} \text{gr/cm}^3$ ).

Many different realistic EOSs have been proposed to date which all produce neutron stars that satisfy the currently available observational constraints (currently, the two main constraints are that the EOS must admit non-rotating neutron stars with gravitational mass of at least  $1.44M_\odot$  and allow rotational periods at least as small as

1.56 ms, see [6, 7]). The proposed EOSs are qualitatively and quantitatively very different from each other. Some are based on relativistic many-body theories while others use nonrelativistic theories with baryon-baryon interaction potentials. A classic collection of early proposed EOSs was compiled by Arnett and Bowers [8], while recent EOSs are described in Salgado et al. [9].

High density equations of state with pion condensation have been proposed by Migdal [10] and Sawyer and Scalapino [11]. The possibility of Kaon condensation is discussed by Brown and Bethe [12] and questioned by Pandharipande et al. [13]. Many authors have examined the possibility of stars composed of strange quark matter and a recent review can be found in [14].

The realistic EOSs are supplied in the form of an energy density vs. pressure table and intermediate values are interpolated. This results in some loss of accuracy because the usual interpolation methods do not preserve thermodynamical consistency. Recently however, Swesty [15] devised a cubic Hermite interpolation scheme that does preserve thermodynamical consistency and the scheme has been shown to indeed produce higher accuracy neutron star models in Nozawa et al. [16].

- **Going further.** A discussion of hybrid stars, that have a mixed-phase region of quark and hadronic matter, can be found in [17]. A study of the relaxation effect in dissipative relativistic fluid theories is presented in [18].

## 2.4 Numerical Schemes

Out of the ten components of the field equations that describe the geometry of a rotating relativistic star, only four are independent and one has the freedom to choose which four components to use. After choosing four field equations, there are different methods one can use to solve them. First models were obtained by Wilson [19] and Bonazzola and Schneider [20]. Here we will review the following methods: Hartle's slow rotation formalism, the Newton-Raphson linearization scheme due to Butterworth and Ipser [21], a scheme using Green's functions by Komatsu et al. [22, 23], a minimal surface scheme due to Wu et al. [24], and two spectral methods by Bonazzola et al. [25, 26]. Below we give a description about each method and its various implementations (codes).

### 2.4.1 Hartle

To  $O(\Omega^2)$  the structure of a star changes only by quadrupole terms and the equilibrium equations become a set of ordinary differential equations. Hartle's [27, 28] method computes rotating stars in this slow-rotation approximation and a review of slowly rotating models has been compiled by Datta [29]. Weber et al. [30], [31] also implement Hartle's formalism to explore the rotational properties of four new EOSs.

Weber and Glendenning [32] attempt to improve on Hartle’s formalism in order to obtain a more accurate estimate of the angular velocity at the mass-shedding limit but their models show large discrepancies compared to corresponding models computed with fully rotating schemes [9]. Thus, Hartle’s formalism cannot be used to compute models of rapidly rotating relativistic stars with sufficient accuracy.

#### 2.4.2 Butterworth and Ipser (BI)

The BI-scheme [21] solves the four field equations following a Newton-Raphson like linearization and iteration procedure. One starts with a nonrotating model and increases the angular velocity in small steps, treating a new rotating model as a linear perturbation of the previously computed rotating model. Each linearized field equation is discretized and the resulting linear system is solved. The four field equations and the hydrostationary equilibrium equation are solved separately and iteratively until convergence is achieved.

The space is truncated at a finite distance from the star and the boundary conditions there are imposed by expanding the metric potentials in powers of  $1/r$ . Angular derivatives are approximated by high-accuracy formulae and models with density discontinuities are treated specially at the surface. An equilibrium model is specified by fixing its rest mass and angular velocity.

The original BI code was used to construct uniform density models and polytropic models [21, 33]. Friedman et al. [34, 35], extend the BI code to obtain a large number of rapidly rotating models based on a variety of realistic EOSs. Lattimer et al. [36] used a code which was also based on the BI scheme to construct rotating stars using recent “exotic” and schematic EOSs, including pion or Kaon condensation and self-bound strange quark matter.

#### 2.4.3 Komatsu, Eriguchi and Hachisu (KEH)

In the KEH scheme [22, 23], the same set of field equations as in BI is used, but the three elliptic-type field equations are converted into integral equations using appropriate Green’s functions. The boundary conditions at large distance from the star are thus incorporated into the integral equations, but the region of integration is truncated at a finite distance from the star. The fourth field equation is an ordinary first-order differential equation. The field equations and the equation of hydrostationary equilibrium are solved iteratively, fixing the maximum energy density and the ratio of the polar radius to the equatorial radius, until convergence is achieved. In [22, 23] and [37] the original KEH code is used to construct uniformly and differentially rotating stars for both polytropic and realistic EOSs.

Cook, Shapiro and Teukolsky (CST) improve on the KEH scheme by introducing a new radial variable which maps the semi-infinite region  $[0, \infty)$  to the closed region

$[0, 1]$ . In this way, the region of integration is not truncated and the model converges to a higher accuracy. Details of the code are presented in [38] and polytropic and realistic models are computed in [39] and [40].

Stergioulas and Friedman (SF) implement their own KEH code following the CST scheme. They improve on the accuracy of the code by a special treatment of the second order radial derivative that appears in the source term of the first-order differential equation for one of the metric functions. This derivative was introducing a numerical error of 1%–2% in the bulk properties of the most rapidly rotating stars computed in the original implementation of the KEH scheme. The SF code is presented in [41] and in [42]. It is available as a public domain code, named `rns`, and can be downloaded from [43].

#### 2.4.4 Wu et al. (WMSHR)

The numerical scheme by Wu et al. [24] implements the minimal surface formalism for rotating axisymmetric space-times [44, 45, 46], in which Einstein’s field equations are equivalent to the minimal surface equations in an abstract Riemannian potential space with a well-defined metric, whose coordinates are the four metric functions of the usual stationary, axisymmetric metric. A finite element technique is used and the system of equations is solved by a Newton-Raphson method. Models based on realistic EOSs are presented in [47]. The WMSHR code has been used to visualize rapidly rotating stars by embedding diagrams and 4D-ray-tracing pictures (see [48] for a review).

#### 2.4.5 Bonazzola et al. (BGSM)

In the BGSM scheme [25], the field equations are derived in the  $3 + 1$  formulation. All four equations describing the gravitational field are of elliptic type. This avoids the problem with the second-order radial derivative in the source term of the ODE used in BI and KEH. The equations are solved using a spectral method, i.e. all functions are expanded in terms of trigonometric functions in both the angular and radial directions and a Fast Fourier Transform (FFT) is used to obtain coefficients. Outside the star a redefined radial variable is used, which maps infinity to a finite distance.

In [49] the code is used to construct a large number of models based on recent EOSs. The accuracy of the computed models is estimated using two general relativistic Virial identities, valid for general asymptotically flat space-times, that were discovered by Gourgoulhon and Bonazzola [50, 51].

While the field equations used in the BI and KEH schemes assume a perfect fluid, isotropic stress-energy tensor, the BGSM formulation makes no assumption about the isotropy of  $T_{ab}$ . Thus, the BGSM code can compute stars with magnetic field, solid crust or solid interior and it can also be used to construct rotating boson

stars.

Since it is based on the  $3 + 1$  formalism, the BGSM code is also suitable for providing high-accuracy, unstable equilibrium models as initial data for an axisymmetric collapse computation.

#### 2.4.6 Bonazzola et al. (BGM-98)

The BGSM spectral method has been improved by Bonazzola et al. [26] allowing for several domains of integration. One of the domain boundaries is chosen to coincide with the surface of the star and a regularization procedure is introduced for the infinite derivatives at the surface (that appear in the density field when stiff equations of state are used). This allows models to be computed that are free of Gibbs phenomena at the surface. The method is also suitable for constructing quasi-stationary models of binary neutron stars.

#### 2.4.7 Direct Comparison of Numerical Codes

The accuracy of the above numerical codes can be estimated, if one constructs exactly the same models with different codes and compares them directly. The first such comparison of rapidly rotating models constructed with the FIP and SF codes is presented by Stergioulas and Friedman in [41]. Rapidly rotating models constructed with several EOS's agree to  $0.1\% - 1.2\%$  in the masses and radii and to better than  $2\%$  in any other quantity that was compared (angular velocity and momentum, central values of metric functions etc.). This is a very satisfactory agreement, considering that the BI code was using relatively few grid points, due to limitations of computing power at the time of its implementation.

In [41], it is also shown that a large discrepancy between certain rapidly rotating models, constructed with the FIP and KEH codes, that was reported by Eriguchi et al. [37], was only due to the fact that a different version of a tabulated EOS was used in [37] than by FIP.

Recently, Nozawa et al. [16] have completed an extensive direct comparison of the BGSM, SF and the original KEH codes, using a large number of models and equations of state. More than twenty different quantities for each model are compared and the relative differences range from  $10^{-3}$  to  $10^{-4}$  or better, for smooth equations of state. The agreement is excellent for soft polytropes, which shows that all three codes are correct and compute the desired models to an accuracy that depends on the number of grid-points used to represent the spacetime.

If one makes the extreme assumption of uniform density, the agreement is at the level of  $10^{-2}$ . In the BGSM code this is due to the fact that the spectral expansion in terms of trigonometric functions cannot accurately represent functions with discontinuous first-order derivatives at the surface of the star. In the KEH and SF codes, the three-point finite-difference formulae cannot accurately

represent derivatives across the discontinuous surface of the star.

The accuracy of the three codes is also estimated by the use of the two Virial identities due to Gourgoulhon and Bonazzola [50, 51]. Overall, the BGSM and SF codes show a better and more consistent agreement than the KEH code with BGSM or SF. This is largely due to the fact that the KEH code does not integrate over the whole spacetime but within a finite region around the star, which introduces some error in the computed models.

- **Going further.** A review of spectral methods in general relativity can be found in [52]. A formulation for nonaxisymmetric, uniformly rotating equilibrium configurations in the second post-Newtonian approximation is presented in [53].

## 2.5 Properties of Equilibrium Models

### 2.5.1 Bulk Properties of Equilibrium Models

Neutron star models constructed with various realistic EOSs have considerably different bulk properties, due to the large uncertainties in the equation of state at high densities. Very compressible (soft) EOSs produce models with small maximum mass, small radius, and large rotation rate. On the other hand, less compressible (stiff) EOSs produce models with a large maximum mass, large radius, and low rotation rate.

The gravitational mass, equatorial radius and rotational period of the maximum mass model constructed with one of the softest EOSs (EOS B) ( $1.63M_{\odot}$ ,  $9.3\text{km}$ ,  $0.4\text{ms}$ ) are a factor of two smaller than the mass, radius and period of the corresponding model constructed by one of the stiffest EOSs (EOS L) ( $3.27M_{\odot}$ ,  $18.3\text{km}$ ,  $0.8\text{ms}$ ). The two models differ by a factor of 5 in central energy density and a factor of 8 in the moment of inertia!

Not all properties of the maximum mass models between proposed EOSs differ considerably. For example, most realistic EOSs predict a maximum mass model with a ratio of rotational to gravitational energy  $T/W$  of  $0.11 \pm 0.02$ , a dimensionless angular momentum  $cJ/GM^2$  of  $0.64 \pm 0.06$  and an eccentricity of  $0.66 \pm 0.04$ , [1]. Hence, between the set of realistic EOSs, some properties are directly related to the stiffness of the EOS while other properties are rather insensitive to stiffness.

Compared to nonrotating stars, the effect of rotation is to increase the equatorial radius of the star and also to increase the mass that can be sustained at a given central energy density. As a result, the mass of the maximum mass rotating model is roughly  $15\% - 20\%$  higher than the mass of the maximum mass nonrotating model, for typical realistic EOSs. The corresponding increase in radius is  $30\% - 40\%$ .

The deformed shape of a rapidly rotating star creates a distortion, away from spherical symmetry, in its gravitational field. Far from the star, the distortion is measured

by the quadrupole-moment tensor  $Q_{ab}$ . For uniformly rotating, axisymmetric and equatorially symmetric configurations, one can define a scalar quadrupole moment  $Q$ , which can be extracted from the asymptotic expansion, at large  $r$ , of the metric function  $\nu$ .

Laarakkers and Poisson [54], numerically compute the scalar quadrupole moment  $Q$  for several equations of state, using the rotating neutron star code **rns** [43]. They find that for fixed gravitational mass  $M$ , the quadrupole moment is given as a simple quadratic fit

$$Q = -a \frac{J^2}{Mc^2}, \quad (13)$$

where  $J$  is the angular momentum of the star and  $a$  is a dimensionless quantity that depends on the equation of state. The above quadratic fit reproduces  $Q$  with a remarkable accuracy. The quantity  $a$  varies between  $a \sim 2$  for very soft EOSs and  $a \sim 8$  for very stiff EOSs, for  $M = 1.4M_\odot$  neutron stars.

For a given zero-temperature EOS, the uniformly rotating equilibrium models form a 2-dimensional surface in the 3-dimensional space of central energy density, gravitational mass and angular momentum [41]. The surface is limited by the nonrotating models ( $J = 0$ ) and by the models rotating at the mass-shedding (Kepler) limit, i.e. at the maximum allowed angular velocity so that the star does not shed mass at the equator. Cook et al. [38, 39, 40] have shown that the model with maximum angular velocity does not coincide with the maximum mass model, but is generally very close to it in central density and mass. Stergioulas and Friedman [41] show that the maximum angular velocity and maximum baryon mass equilibrium models are also distinct. The distinction becomes significant in the case where the EOS has a large phase transition near the central density of the maximum mass model, otherwise the models of maximum mass, baryon mass, angular velocity and angular momentum can be considered to coincide for most purposes.

### 2.5.2 An Empirical Formula for the Kepler Velocity

In the Newtonian limit the maximum angular velocity of uniformly rotating polytropic stars is,  $\Omega_{max} \simeq (2/3)^{3/2}(GM/R^3)^{1/2}$  (see [150]). For relativistic stars, the empirical formula [55, 35, 56]

$$\Omega_{max} = 0.67 \sqrt{\frac{GM_{max}}{R_{max}^3}}, \quad (14)$$

gives the maximum angular velocity in terms of the mass and radius of the maximum mass *nonrotating* model with an accuracy of 5% – 7%, without actually having to construct rotating models.

The empirical formula results from universal proportionality relations that exist between the mass and radius of the maximum mass rotating model and those of the

maximum mass nonrotating model for the same EOS. La-sota et al. [57] find that, for most EOSs, the coefficient in the empirical formula is an almost linear function of the parameter

$$\chi_s = \frac{2GM_{max}}{R_{max}c^2}, \quad (15)$$

When this relation is taken into account in the empirical formula, it reproduces the exact values with a relative error of only 1.5%.

Weber and Glendenning [30, 32], try to reproduce analytically the empirical formula in the slow rotation approximation but the formula they obtain involves the mass and radius of the maximum mass rotating configuration, which is different from what is involved in (14).

### 2.5.3 The Upper Limit on Mass and Rotation

The maximum mass and minimum period of rotating relativistic stars computed with realistic EOSs from the Arnett and Bowers collection [8] are about  $3.3M_\odot$  (EOS L) and 0.4ms (EOS B), while  $1.4M_\odot$  neutron stars, rotating at the Kepler limit, have a rotational periods between 0.53ms (EOS B) and 1.7ms (EOS M) [40]. The maximum, accurately measured, neutron star mass is currently  $1.44M_\odot$ , but there are also indications for  $2.0M_\odot$  neutron stars [58]. The minimum observed pulsar period is 1.56ms [7], which is close to the experimental sensitivity of recent pulsar searches (an ongoing experiment is designed to detect sub-millisecond pulsars, if they exist [59]).

In principle, neutron stars with maximum mass or minimum period could exist, if they are born as such in a core collapse, or if they accrete the right amount of matter and angular momentum during an accretion-induced spin-up phase. Such a phase could also follow the creation of an  $1.4M_\odot$  neutron star during the accretion induced collapse of a white dwarf.

In reality, only a very small fraction, if any, of neutron stars will be close to the maximum mass or minimum period limit. In addition, rapidly rotating nascent neutron stars are subject to a nonaxisymmetric instability, which lowers their initial rotation rate and neutron stars with a strong magnetic field have their rotation rate limited by the Kepler velocity at their Alfvén radius, where the accretion pressure balances the magnetospheric pressure [7].

- **Going further.** A recent review by J. L. Friedman on the upper limit on rotation of relativistic stars can be found in [60].

### 2.5.4 The Upper Limit on Mass and Rotation Set by Causality

Current proposed EOSs are reliable only to about twice nuclear density and result in very different values for the maximum mass and minimum period of neutron stars. If one is interested in obtaining upper limits on the mass

and rotation rate, independent of the proposed EOSs, one has to rely on fundamental physical principles.

Instead of using realistic EOSs, one constructs a set of artificial EOSs that satisfy only a minimal set of physical constraints, which represent what we know about the equation of state of matter with high confidence. One then searches among all these EOSs to obtain the one that gives the maximum mass or minimum period. The minimal set of constraints that have been used in such searches are that

1. the high density EOS matches to the known low density EOS at some matching energy density  $\epsilon_m$ ,
2. the matter at high densities satisfies the causality constraint (the speed of sound is less than the speed of light).

In relativistic perfect fluids, the speed of sound is the characteristic velocity of the fluid evolution equations and the causality constraint translates into the requirement

$$dp/d\epsilon \leq 1. \quad (16)$$

(see e.g. Geroch and Lindblom [61]). It is assumed that the fluid will still behave as a perfect fluid when it is perturbed from equilibrium.

For nonrotating stars, Rhoades and Ruffini showed that the EOS that satisfies the above two constraints and yields the maximum mass consists of a high density region as stiff as possible (i.e. at the causal limit,  $dp/d\epsilon = 1$ ), that matches directly to the known low density EOS. For a chosen matching density  $\epsilon_m$ , they computed a maximum mass of  $3.2M_\odot$ . However, this is not the theoretically maximum mass of nonrotating neutron stars, as is often quoted in the literature. Hartle and Sabbadini [62] point out that  $M_{max}$  is sensitive to the matching energy density and Hartle [63] computes  $M_{max}$  as a function of  $\epsilon_m$ .

$$M_{max} = 4.8 \left( \frac{2 \times 10^{14} \text{gr/cm}^3}{\epsilon_m} \right)^{1/2} M_\odot. \quad (17)$$

In the case of rotating stars, Friedman and Ipser [64] assume that the absolute maximum mass is obtained by the same EOS as in the nonrotating case and compute  $M_{max}$  as a function of matching density, assuming the BPS EOS holds at low densities. Stergioulas and Friedman [41] recompute  $M_{max}^{rot}$  for rotating stars using the more recent FPS EOS at low densities, obtaining very nearly the same result

$$M_{max}^{rot} = 6.1 \left( \frac{2 \times 10^{14} \text{gr/cm}^3}{\epsilon_m} \right)^{1/2} M_\odot, \quad (18)$$

where,  $2 \times 10^{14} \text{gr/cm}^3$  is roughly nuclear saturation density for the FPS EOS.

A first estimate of the absolute minimum period of uniformly rotating, gravitationally bound stars was computed by Glendenning [65] by constructing nonrotating

models and using the empirical formula (14) to estimate the minimum period. Koranda, Stergioulas and Friedman [66] improve on Glendenning's results by constructing accurate rapidly rotating models and show that Glendenning's results are accurate to within the accuracy of the empirical formula.

Furthermore, they show that the EOS satisfying the minimal set of constraints and yielding the minimum period star consists of a high density region at the causal limit, which is matched to the known low density EOS through an intermediate constant pressure region (that would correspond to a first-order phase transition). Thus, the EOS yielding absolute minimum period models is as stiff as possible at the central density of the star (to sustain a large enough mass) and as soft as possible in the crust, in order to have the smallest possible radius (and rotational period).

The absolute minimum period of uniformly rotating stars is an (almost linear) function of the maximum observed mass of nonrotating neutron stars

$$P_{min} = 0.28 \text{ms} + 0.2(M_{max}^{nonrot.} - 1.44M_\odot), \quad (19)$$

and is rather insensitive to the matching density  $\epsilon_m$  (the above result was computed for a matching number density of  $0.1 \text{fm}^{-3}$ ).

In [66], it is also shown that an absolute limit on the minimum period exists even without requiring that the EOS matches to a known low density EOS (this is not true for the limit on the maximum mass). Thus, using causality as the only constraint on the EOS,  $P_{min}$  is lowered by only 3%, which shows that the currently known part of the nuclear EOS plays a negligible role in determining the absolute upper limit on the rotation of uniformly rotating, gravitationally bound stars.

## 2.5.5 Spin-Up Prior to Collapse

Since rotation increases the mass that a neutron star of given central density can support, there exist sequences of neutron stars with constant baryon number that have no nonrotating member. Such sequences are called supra-massive as opposed to normal sequences that do have a nonrotating member. A nonrotating star can become supra-massive by accreting matter and spinning-up to large rotation rates; in another scenario, neutron stars could be born supra-massive after a core collapse. A supra-massive star evolves along a sequence of constant baryon mass, slowly losing angular momentum. Eventually, the star reaches a point where it becomes unstable to axisymmetric perturbations and collapses to a black hole. The instability grows on a secular timescale, in the sense that it is limited by the time required for viscosity to redistribute the star's angular momentum. This timescale is comparable with the spin-up time following a glitch [1].

Cook et al. [38, 39, 40] have discovered that a supra-massive star approaching the axisymmetric instability, will actually spin-up before collapse, even though it loses

angular momentum. This, potentially observable, effect is independent of the equations of state and it is more pronounced for rapidly rotating massive stars. In a similar phenomenon, normal stars can spin-up by loss of angular momentum near the Kepler limit, if the equation of state is extremely stiff or extremely soft.

### 2.5.6 Rotating Magnetized Neutron Stars

The presence of a magnetic field was ignored in the models of rapidly rotating relativistic stars that were considered in the previous sections. The reason is that the observed surface dipole magnetic field strength of pulsars ranges between  $B = 10^8$  G and  $B = 2 \times 10^{13}$  G. These values of  $B$  imply a magnetic field energy density that is too small compared to the energy density of the fluid, to significantly affect the structure of a neutron star. However, one cannot exclude the existence of neutron stars with higher magnetic field strengths or the possibility that neutron stars are born with much stronger magnetic fields, which then decay to the observed values (Of course there are also many arguments against magnetic field decay in neutron stars [6]). In addition, even though moderate magnetic field strengths do not alter the bulk properties of neutron stars, they may have an effect on the damping or growth rate of various perturbations of an equilibrium star, affecting its stability. For these reasons, a fully relativistic description of magnetized neutron stars is desirable and, in fact, Bocquet et al. [67] achieved the first numerical computation of such configurations. Here we give a brief summary of their work:

A magnetized relativistic star in equilibrium can be described by the coupled Einstein-Maxwell field equations for stationary, axisymmetric rotating objects with internal electric currents. The stress-energy tensor includes the electromagnetic energy density and is non-isotropic (in contrast to the isotropic perfect fluid stress-energy tensor). The equilibrium of the matter is given not only by the balance between the gravitational force and the pressure gradient, but the Lorentz force due to the electric currents also enters the balance. For simplicity, Bocquet et al. consider only poloidal magnetic fields, which preserve the circularity of the space-time. Also, they only consider stationary configurations, which excludes magnetic dipole moments non-aligned with the rotation axis, since in that case the star emits electromagnetic and gravitational waves. The assumption of stationarity implies that the fluid is necessarily rigidly rotating (if the matter has infinite conductivity) [25]. Under these assumptions, the electromagnetic field tensor  $F^{ab}$  is derived from a potential 1-form  $A_a$  with only two non-vanishing components,  $A_t$  and  $A_\phi$ , which are given by a scalar Poisson and a vector Poisson equation respectively. Thus, the two equations describing the electromagnetic field are of similar type as the four field equations that describe the gravitational field.

The construction of magnetized models with  $B < 10^{13}$

G confirms that magnetic fields of this strength have a negligible effect on the structure of the star. However, if one increases the strength of the magnetic field above  $10^{14}$  G, one observes significant effects, such as a flattening of the star. The magnetic field cannot be increased indefinitely, but there exists a maximum value of the magnetic field strength, of the order of  $10^{17}$  G, for which the magnetic field pressure at the center of the star equals the fluid pressure. Above this value, the fluid pressure decreases more rapidly away from the center along the symmetry axis, than the magnetic pressure. Instead of pressure, there is tension along the symmetry axis and no stationary configuration can exist.

The shape of a strongly magnetized star is flattened because the Lorentz forces exerted by the E/M field on the fluid act as centrifugal forces. A star with a magnetic field near the maximum value for stationary configurations, displays a pinch along the symmetry axis, because there, the magnetic pressure exceeds the fluid pressure. The maximum fluid density inside the star is not attained at the center, but away from it. The presence of a strong magnetic field also allows a maximum mass configuration with larger  $M_{max}$  than for the same EOS with no magnetic field and this is in analogy with the increase of  $M_{max}$  induced by rotation. For nonrotating stars, the increase in  $M_{max}$ , due to a strong magnetic field, is 13% – 29%, depending on the EOS. Following the increase in mass, the maximum allowed angular velocity for a given EOS also increases in the presence of a magnetic field.

Bocquet et al. are planning to use their code in the study of two types of possible instabilities in magnetized neutron stars, i) a pure E/M instability towards another electric current/magnetic field distribution of lower energy and ii) a nonaxisymmetric instability for rapidly rotating models, which would be the analog of a Jacobi-type transition in non-magnetized stars. In perfect fluid models with a magnetic field, one would also expect a CFS-instability driven by electromagnetic waves.

### 2.5.7 Rapidly Rotating Proto-Neutron Stars

Following the gravitational collapse of a massive stellar core, a proto-neutron star (PNS) is born. Initially it has a large radius of about 100km and a temperature of 50-100MeV. The PNS may be born with a large rotational kinetic energy and initially it will be differentially rotating. Due to the violent nature of the gravitational collapse, the PNS pulsates heavily, emitting significant amounts of gravitational radiation. After a few hundred pulsational periods, bulk viscosity will damp the pulsations significantly. Rapid cooling due to deleptonization transforms the PNS to a hot neutron star of  $T \sim 10$ MeV shortly after its formation. In addition, viscosity reduces the differential rotation to a nearly uniform rotation on a timescale of seconds [2] and the neutron star becomes quasi-stationary. Since the details of the PNS evolution determine the exact properties of the resulting cold NSs,



proto-neutron stars must be modeled realistically in order to understand the structure of cold neutron stars.

Hashimoto et al. [68] and Goussard et al. [69] recently constructed fully relativistic models of rapidly rotating, hot proto-neutron stars. The authors use finite-temperature EOSs [70, 71], to model the interior of PNSs. Important parameters, which determine the local state of matter but are largely unknown, are the lepton fraction  $Y_l$  and the temperature profile. Hashimoto et al. consider only the limiting case of zero lepton fraction  $Y_l = 0$  and classical isothermality, while Goussard et al. consider several non-zero values for  $Y_l$  and two different limiting temperature profiles - a constant entropy profile and a relativistic isothermal profile. In both [68] and [70], differential rotation is neglected to a first approximation.

The construction of numerical models with the above assumptions shows that, due to the high temperature and the presence of trapped neutrinos, PNSs have a significantly larger radius than cold NSs. These two effects give the PNS an extended envelope which, however, contains only roughly 0.1% of the total mass of the star. This outer layer cools more rapidly than the interior and becomes transparent to neutrinos, while the core of the star remains hot and neutrino opaque for a longer time. The two regions are separated by the “neutrino sphere”.

Compared to the  $T = 0$  case, an isothermal EOS with temperature of 25MeV has a maximum mass model of only slightly larger mass. In contrast, an isentropic EOS with a nonzero trapped lepton number features a maximum mass model that has a considerably lower mass than the corresponding model in the  $T = 0$  case and a stable PNS transforms to a stable neutron star. If, however, one considers the hypothetical case of a large amplitude phase transition which softens the cold EOS (such as a Kaon condensate), then  $M_{max}$  of cold neutron stars is lower than  $M_{max}$  of PNSs and a stable PNS with maximum mass will collapse to a black hole after the initial cooling period. This scenario of delayed collapse of nascent neutron stars has been proposed by Brown and Bethe [12] and investigated by Baumgarte et al. [72].

An analysis of radial stability of PNSs [73] shows that, for hot PNSs, the maximum angular velocity star almost coincides with the maximum mass star, as is also the case for cold EOSs.

Because of their increased radius, PNSs have a different mass-shedding limit than cold NSs. For an isothermal profile, the mass-shedding limit proves to be sensitive to the exact location of the neutrino sphere. For the EOSs considered in [68] and [69] PNSs have a maximum angular velocity that is considerably less than the maximum angular velocity allowed by the cold EOS. Stars that have nonrotating counterparts (i.e. that belong to a normal sequence) contract and speed up while they cool down. The final star with maximum rotation is thus closer to the mass-shedding limit of cold stars than was the hot PNS with maximum rotation. Surprisingly, stars belonging to a supra-massive sequence exhibit the opposite behavior.

If one assumes that a PNS evolves without losing angular momentum or accreting mass, then a cold neutron star produced by the cooling of a hot PNS has a smaller angular velocity than its progenitor. This purely relativistic effect was pointed out in [68] and confirmed in [69]. It should be noted here, that a small amount of differential rotation significantly affects the mass-shedding limit, allowing more massive stars to exist than uniform rotation allows. Taking differential rotation into account, a more recent study by Goussard et al. [2] suggests that proto-neutron stars created in a gravitational collapse cannot spin faster than 1.7 ms.

### 3 Oscillations and Stability

The study of oscillations of relativistic stars has the potential of yielding important information about both the bulk properties and the composition of the interior of the star i.e. about the equation of state of matter at very high densities, in about the same way that helioseismology is providing us with information about the interior of the Sun. In a neutron star - accretion disk system, the star-disk interaction can drive oscillations and one of the possible explanations for kHz quasi-periodic oscillations recently discovered in several X-ray sources are neutron star pulsations [74] (for an early proposal that such oscillations may be observable, see [75]).

Neutron star pulsations may be a detectable source of gravitational radiation. The pulsations can be excited after a core collapse or during the final stages of a neutron star binary system coalescence. Rapidly rotating neutron stars are unstable to the emission of detectable gravitational waves for a short time after their formation. The identification of gravitational waves produced by a neutron star can lead to the determination of its mass and radius and several such determinations can help reconstruct the equation of state of matter at very high energy densities [76].

The oscillations of relativistic stars are actually a non-linear phenomenon and their numerical computation would require a full 3-D relativistic hydrodynamics code, which is not yet available. However, apart from the initial oscillations following core collapse, the oscillations of an equilibrium star are of small magnitude compared to its radius and it will suffice to approximate them as linear perturbations. Such perturbations can be described in two equivalent ways. In the Lagrangian approach, one studies the changes in a given fluid element as it oscillates about its equilibrium position. In the Eulerian approach, one studies the change in fluid variables at a fixed point in space. Both approaches have their strengths and weaknesses.

In the Newtonian limit, the Lagrangian approach has been used to develop variational principles [77, 78] but the Eulerian approach proved to be more suitable for numerical computations of mode frequencies and eigenfunctions

[79, 80, 81, 82, 83]. Clement [84] used the Lagrangian approach to obtain axisymmetric normal modes of rotating stars, while nonaxisymmetric solutions were obtained in the Lagrangian approach by Imamura et al. [85] and in the Eulerian approach by Managan [80] and Ipser and Lindblom [81].

### 3.1 Quasi-Normal Modes of Oscillation

The spacetime of a nonrotating star is static and spherically symmetric. A general linear perturbation can be written as a sum of quasi-normal modes that are characterized by the indices  $(l, m)$  of the spherical harmonic  $Y_l^m$  and have angular and time-dependence of the form

$$\delta Q \sim f(r) Y_l^m(\cos\theta) e^{i\omega_p t}, \quad (20)$$

where  $Q$  is a scalar unperturbed quantity,  $\omega_p$  is the angular frequency of the mode, as measured by a distant inertial observer and  $f(r)$  represents the radial dependence of the perturbation. Normal modes of nonrotating stars are degenerate in  $m$  and it suffices to study the axisymmetric ( $m = 0$ ) case.

The perturbation of the metric,  $\delta g_{ab}$ , can be expressed in terms of spherical, vector and tensor harmonics. These are either of "polar" or "axial" parity. Here, parity is defined as the change in sign under a combination of reflection in the equatorial plane and rotation by  $\pi$ . A polar perturbation has parity  $(-1)^l$ , while an axial perturbation has parity  $(-1)^{l+1}$ . Because of the spherical background, the polar and axial perturbations of a nonrotating star are completely decoupled.

A normal mode solution satisfies the perturbed gravitational field equations

$$\delta(G^{ab} - 8\pi T^{ab}) = 0, \quad (21)$$

and the perturbation of the conservation of the stress-energy tensor

$$\delta(\nabla_a T^{ab}) = 0. \quad (22)$$

For given  $(l, m)$ , a solution exists for any value of the eigenfrequency  $\omega_p$  and it consists of ingoing- and outgoing-wave parts. Outgoing modes are defined by the discrete set of eigenfrequencies for which there are no incoming waves at infinity. These are the modes that will be excited in various astrophysical situations.

The main modes of pulsation that are known to exist in relativistic stars have been classified as follows ( $f_0$  and  $\tau_0$  are typical frequencies and damping times of the most important modes in the nonrotating limit):

#### 1. Polar fluid modes

Are slowly damped modes analogous to the Newtonian fluid pulsations:

- $f$ (undamental)-mode: surface mode due to the interface between the star and its surroundings ( $f_0 \sim 2\text{kHz}$ ,  $\tau_0 < 1\text{sec}$ ),

- $p$ (ressure)-modes: nearly radial ( $f_0 > 4\text{kHz}$ ,  $\tau_0 > 1\text{s}$ ),
- $g$ (ravity modes): nearly tangential, only exist for finite temperature stars ( $f_0 < 500\text{Hz}$ ,  $\tau_0 > 5\text{s}$ ).

#### 2. Axial fluid modes

- $r$ (otation) modes: degenerate at zero-frequency for nonrotating stars. In a rotating star, generically unstable. Frequencies from zero to kHz, growth times inversely proportional to a high power of the star's angular velocity.

#### 3. Polar and axial spacetime modes

- $w$ (ave)-modes: Analogous to the quasi-normal modes of a black hole. High frequency, strongly damped modes ( $f_0 > 6\text{kHz}$ ,  $\tau_0 \sim 0.1\text{msec}$ ).

For a more detailed description of various modes see [86, 87, 88, 89, 90].

### 3.2 Effect of Rotation on Quasi-Normal Modes

In a continuous sequence of rotating stars, a quasi-normal mode of index  $l$  is defined as the mode which, in the nonrotating limit, reduces to the quasi-normal mode of the same index  $l$ . Rotation has several effects on the modes of a previously nonrotating star:

1. The degeneracy in the index  $m$  is removed and a non-rotating mode of index  $l$  is split into  $2l + 1$  different  $(l, m)$  modes.
2. Prograde ( $m < 0$ ) modes are now different than retrograde ( $m > 0$ ) modes.
3. A rotating "polar"  $l$ -mode consists of a sum of purely polar and purely axial terms [42]

$$P_l^{rot} \sim \sum_{l'=0}^{\infty} (P_{l+2l'} + A_{l+2l'\pm 1}), \quad (23)$$

that is, rotation couples a polar  $l$ -term to an axial  $l \pm 1$  term (the coupling to the  $l + 1$  term is, however, strongly favored over the coupling to the  $l - 1$  term [91]). Similarly, for a rotating "axial" mode,

$$A_l^{rot} \sim \sum_{l'=0}^{\infty} (A_{l+2l'} + P_{l+2l'\pm 1}), \quad (24)$$

4. Frequencies and damping times are shifted. In general, frequencies (in the inertial frame) of prograde modes increase, while those of retrograde modes decrease with increasing rate of rotation.

In rotating stars, quasi-normal modes of oscillation have only been studied in the slow-rotation limit, in the post-Newtonian and in the Cowling Approximations. The solution of the fully-relativistic perturbation equations for a rapidly rotating star is still a very challenging task and only recently they have been solved for zero- frequency (neutral) modes [42, 92].

- **Going further.** The equations that describe oscillations of the solid crust of a rapidly rotating relativistic star are derived by Priou in [93]. The effects of superfluid hydrodynamics on the oscillations of neutron stars are investigated by Lindblom and Mendell in [94].

### 3.3 Axisymmetric Perturbation

Along a sequence of nonrotating relativistic stars with increasing central energy density, there is always a model for which the mass becomes maximum. The maximum mass turning point marks the onset of a secular instability in the fundamental axisymmetric pulsation mode of the star.

Applying the turning point theorem provided by Sorkin [95], Friedman Ipser and Sorkin [96] show that in the case of rotating stars the secular axisymmetric instability sets in when the mass becomes maximum along a sequence of constant angular momentum. An equivalent criterion is provided by Cook et al. [38] : the secular axisymmetric instability sets in when the angular momentum becomes minimum along a sequence of constant rest mass.

The instability develops on a timescale that is limited by the time required for viscosity to redistribute the star's angular momentum. This timescale is long compared to the dynamical timescale and comparable to the spin-up time following a pulsar glitch. When it becomes secularly unstable, a star evolves in a quasi-stationary fashion until it encounters the dynamical instability and collapses to a black hole. Thus, the onset of the secular instability to axisymmetric perturbations separates stable neutron stars from neutron stars that will collapse to a black hole.

Goussard et al. [69] extend the stability criterion to hot protoneutron stars with nonzero total entropy. In this case, the loss of stability is marked by the configuration with minimum angular momentum along a sequence of both constant rest mass and total entropy.

In the nonrotating limit, Gondek et al. [73] compute frequencies and eigenfunctions of axisymmetric pulsations of hot proto-neutron stars and verify that the secular instability sets in at the maximum mass turning point, as is the case for cold neutron stars.

- **Going further** The stabilization of a relativistic star, that is marginally stable to axisymmetric perturbations, by an external gravitational field, is discussed in [97].

## 3.4 Nonaxisymmetric Perturbations

### 3.4.1 Nonrotating Limit

For a spherical star, it suffices to study the  $m = 0$  axisymmetric modes of pulsation, since the  $m \neq 0$  modes can be obtained by a rotation of the coordinate system.

Thorne, Campolattaro and Price, in a series of papers [98, 99, 100], initiated the computation of nonradial modes by formulating the problem in the Regge-Wheeler (RW) gauge [101] and numerically computing nonradial modes for a number of neutron star models. A variational method for obtaining eigenfrequencies and eigenfunctions has been constructed by Detweiler and Ipser [102]. Lindblom and Detweiler [103] explicitly reduced the system of equations to four first-order ordinary differential equations and obtained more accurate eigenfrequencies and damping times for a larger set of neutron star models. They later realized that their system of equations is sometimes singular inside the star and obtained an improved set of equations which is free of this singularity [104].

Chandrasekhar and Ferrari [91] express the nonradial pulsations in terms of a fifth-order system in a diagonal gauge, which is independent of fluid variables. They thus reformulate the problem in a way analogous to the scattering of gravitational waves off a black hole. Ipser and Price [105] show that in the RW gauge, nonradial pulsations can be described by a system of two second-order equations, which can also be independent of fluid variables. In addition, they find that the diagonal gauge of Chandrasekhar and Ferrari has a remaining gauge freedom which, when removed, also leads to a fourth-order system of equations [106].

In order to locate purely outgoing-wave modes, one has to be able to distinguish the outgoing-wave part from the ingoing-wave part at infinity. In the Thorne et al. and Lindblom and Detweiler schemes, this is achieved using analytic approximations of the solution at infinity.

$W$ -modes pose a more challenging numerical problem because they are strongly damped and the techniques used for  $f$  and  $p$  modes fail to distinguish the outgoing-wave part, but Andersson, Kokkotas and Schutz [107], successfully combine a redefinition of variables with a complex-coordinate integration method, obtaining highly accurate complex frequencies for  $w$  modes. In this method, the ingoing and outgoing solutions are separated by numerically calculating their analytic continuations to a place in the complex-coordinate plane, where they have comparable amplitudes. Since this approach is purely numerical, it could prove to be suitable for the computation of quasi-normal modes in rotating stars, where analytic solutions at infinity are not available.

The non-availability of asymptotic solutions at infinity in the case of rotating stars is one of the major difficulties for computing outgoing modes in rapidly rotating relativistic stars. A new development that may help to overcome this problem, at least to an acceptable approx-

imation, is presented in [108] by Lindblom, Mendell and Ipser.

The authors obtain approximate near-zone boundary conditions for the outgoing modes that replace the outgoing-wave condition at infinity and that enable one to compute the eigenfrequencies with very satisfactory accuracy. First, the pulsation equations of polar modes in the Regge-Wheeler gauge are reformulated as a set of two second-order radial equations for two potentials - one corresponding to fluid perturbations and the other to the perturbations of the spacetime. The equation for the space-time perturbation reduces to a scalar wave equation at infinity and to Laplace's equation for zero-frequency solutions. From these, an approximate boundary condition for outgoing modes is constructed and imposed in the near zone of the star (in fact on its surface) instead at infinity. For polytropic models, the near-zone boundary condition yields  $f$ -mode eigenfrequencies with real parts accurate to 0.01% – 0.1% and imaginary parts with accuracy at the 10% – 20% level, for the most relativistic stars. If the near zone boundary condition can be applied to the oscillations of rapidly rotating stars, the resulting frequencies and damping times should have comparable accuracy.

### 3.4.2 Slow Rotation Approximation

The slow rotation approximation has proven to be useful for obtaining a first estimate of the effect of rotation on the pulsations of relativistic stars. To lowest order in rotation, a polar  $l$ -mode of an initially nonrotating star couples to an axial  $l \pm 1$  mode in the presence of rotation. Conversely, an axial  $l$ -mode couples to a polar  $l \pm 1$  mode [91].

The equations of nonaxisymmetric perturbations in the slow-rotation limit and in the Regge-Wheeler gauge are derived by Kojima in [109, 110], where the complex frequencies  $\sigma = \sigma_R + i\sigma_I$  for the  $l = m$  modes of various polytropes are computed. For counterrotating modes, both  $\sigma_R$  and  $\sigma_I$  decrease, tending to zero, as the rotation rate increases (when  $\sigma$  passes through zero, the star becomes unstable to the CFS-instability). Extrapolating  $\sigma_R$  and  $\sigma_I$  to higher rotation rates, Kojima finds a large discrepancy between the points where  $\sigma_R$  and  $\sigma_I$  go through zero. This shows that the slow rotation formalism cannot accurately determine the onset of the CFS-instability of polar modes in rapidly rotating neutron stars.

In [111], it is shown that, for slowly rotating stars, the coupling between polar and axial modes affects the frequency of pulsation only to second order in rotation, so that, in the slow rotation approximation, to  $O(\Omega)$ , the coupling can be neglected when computing frequencies.

The slow rotation approximation has also been used recently in the study of the  $r$ -mode instability [139, 112].

### 3.4.3 Post-Newtonian Approximation

A first step towards the solution of the perturbation equations in full relativity has been taken by Cutler and Lindblom [113, 114, 115] who obtain frequencies for the  $l = m$   $f$ -modes in rotating stars in the first post-Newtonian (1-PN) approximation. The perturbation equations are derived in the post-Newtonian formalism of Gunnarsen [116], i.e. the equations are separated into equations of consistent order in  $1/c$ .

Cutler and Lindblom show that in this scheme, the perturbation of the 1-PN correction of the four-velocity of the fluid can be obtained analytically in terms of other variables, similarly to what is done for the perturbation in the four-velocity in the Newtonian Ipser-Managan scheme. The perturbation in the 1-PN corrections are obtained by solving an eigenvalue problem, which consists of three second order equations, with the 1-PN correction to the eigenfrequency of a mode,  $\Delta\omega$ , as the eigenvalue.

Cutler and Lindblom obtain a formula that yields  $\Delta\omega$  if one knows the 1-PN stationary solution and the solution to the Newtonian perturbation equations. Thus, the frequency of a mode in the 1-PN approximation can be obtained without actually solving the 1-PN perturbation equations numerically. The 1-PN code was checked in the nonrotating limit and it was found to reproduce the exact general relativistic frequencies for stars with  $M/R = 0.2$  obeying an  $N = 1$  polytropic EOS with an accuracy of 3% – 8%.

Along a sequence of rotating stars, the frequency of a mode is commonly described by the ratio of the frequency of the mode in the comoving frame to the frequency of the mode in the nonrotating limit. For an  $N = 1$  polytrope and for  $M/R = 0.2$ , this frequency ratio is reduced by as much as 12% in the 1-PN approximation compared to its Newtonian counterpart (for the fundamental  $l = m$  modes) which is representative of the effect that general relativity has on the frequency of quasi-normal modes in rotating stars.

### 3.4.4 Cowling Approximation

In several situations, the frequency of pulsations in relativistic stars can be estimated even if one completely neglects the perturbation in the gravitational field, i.e. if one sets  $\delta g_{ab} = 0$  in the perturbation equations [117]. In this approximation, the pulsations are described only by the perturbation in the fluid variables and the scheme works quite well for  $f$ ,  $p$  and  $r$ -modes [118]. A different version of the Cowling approximation, in which  $\delta g_{tr}$  is kept nonzero in the perturbation equations, works better for  $g$ -modes [119].

Yoshida and Kojima [120], examine the accuracy of the relativistic Cowling approximation in slowly rotating stars. The first-order correction to the frequency of a mode depends only on the eigenfrequency and eigenfunctions of the mode in the absence of rotation and on

the angular velocity of the star. The eigenfrequencies of  $f$ ,  $p_1$  and  $p_2$  modes for slowly rotating stars with  $M/R$  between 0.05 and 0.2 are computed (assuming polytropic EOSs with  $N = 1$  and  $N = 1.5$  and compared to their counterparts in the slow-rotation approximation).

For the  $l = 2$   $f$ -mode, the relative error in the eigenfrequency because of the Cowling approximation is 30% for less relativistic stars ( $M/R = 0.05$ ) and about 15% for stars with  $M/R = 0.2$  and the error decreases for higher  $l$ -modes. For the  $p_1$  and  $p_2$  modes the relative error is similar in magnitude but it is smaller for less relativistic stars. Also, for  $p$ -modes, the Cowling approximation becomes more accurate for increasing radial mode number.

As an application, Yoshida and Eriguchi [121] use the Cowling approximation to estimate the onset of the CFS instability in rapidly rotating relativistic stars.

## 3.5 Nonaxisymmetric Instabilities

### 3.5.1 Introduction

Rotating cold neutron stars, detected as pulsars, have a remarkably stable rotation period. But, at birth, or during accretion, rapidly rotating neutron stars can be subject to various nonaxisymmetric instabilities, which will affect the evolution of their rotation rate

If a protoneutron star has a sufficiently high rotation rate (larger than  $T/W \sim 0.27$  for uniformly rotating, constant density Maclaurin spheroids), it will be subject to a dynamical instability driven by hydrodynamics and gravity. Through the  $l = 2$  mode, the instability will deform the star into a bar shape. This highly nonaxisymmetric configuration will emit strong gravitational waves with frequencies in the kHz regime. The development of the instability and the resulting waveform have been computed numerically in the context of Newtonian gravity and hydrodynamics by Houser et al. [122].

At lower rotation rates, the star can become unstable to secular nonaxisymmetric instabilities, driven by gravitational radiation or viscosity. Gravitational radiation drives a nonaxisymmetric instability when a mode that is retrograde with respect to the star appears as prograde to a distant observer, via the Chandrasekhar-Friedman-Schutz (CFS) mechanism [123, 78]: A mode that is retrograde in the corotating frame has negative angular momentum, because the perturbed star has less angular momentum than the unperturbed one. If, to a distant observer, the mode appears prograde, it removes positive angular momentum from the star and thus the angular momentum of the mode becomes increasingly negative.

The instability evolves on a secular timescale, during which the star loses angular momentum via the emitted gravitational waves. When the star rotates slow enough, the mode becomes stable and the instability proceeds on the longer timescale of the next unstable mode, unless it is suppressed by viscosity.

Neglecting viscosity, the CFS-instability is generic in

rotating stars for both polar and axial modes. For polar modes, the instability occurs only above some critical angular velocity, where the frequency of the mode goes through zero in the inertial frame. The critical angular velocity is smaller for increasing mode number  $l$ . Thus, there will always be a high enough mode number  $l$ , for which a slowly rotating star will be unstable. Axial modes are generically unstable in all rotating stars, since the mode has zero frequency in the inertial frame when the star is nonrotating [139, 140].

The shear and bulk viscosity of neutron star matter is able to suppress the growth of the CFS-instability except when the star passes through a certain temperature window. In Newtonian gravity, it appears that the polar mode CFS-instability can occur only in nascent neutron stars that rotate close to the mass-shedding limit [82, 83, 124, 125, 126], but the determination of neutral  $f$ -modes in full relativity [42, 92] shows that relativity enhances the instability, allowing it to occur in stars with smaller rotation rates than previously thought.

- **Going further.** A new numerical method for the analysis of the ergoregion instability in relativistic stars, which may also be used for the analysis of non-axisymmetric instabilities, is presented by Yoshida and Eriguchi in [128].

### 3.5.2 CFS-Instability of Polar Modes

The existence of the CFS-instability in rotating stars was first demonstrated by Chandrasekhar [123] in the case of the  $l = 2$  mode in uniformly rotating, constant density Maclaurin spheroids. Friedman and Schutz [78], show that this instability also appears in compressible stars and that all rotating self-gravitating perfect fluid configurations are generically unstable to the emission of gravitational waves. In addition, they find that a nonaxisymmetric mode becomes unstable when its frequency vanishes in the inertial frame. Thus, zero-frequency outgoing-modes in rotating stars are neutral (marginally stable).

In the Newtonian limit, neutral modes have been determined for several polytropic EOSs [85, 80, 81, 125]. The instability first sets in through  $l = m$  modes. Modes with larger  $l$  become unstable at lower rotation rates but viscosity limits the interesting ones to  $l \leq 5$ . For an  $N = 1$  polytrope, the critical values of  $T/W$  for the  $l = 3, 4$  and 5 modes are 0.079, 0.058 and 0.045 respectively and these values become smaller for softer polytropes.

The  $l = m = 2$  "bar" mode behaves considerably different than the other modes. Its critical  $T/W$  ratio is 0.14 and it is almost independent of the polytropic index. Since soft EOSs cannot produce models with high  $T/W$  values, the bar mode instability appears only for stiff Newtonian polytropes of  $N \leq 0.808$  [129, 130]. In addition, the viscosity driven bar mode appears at the same critical  $T/W$  ratio as the bar mode driven by gravitational radiation (we will see later that this is no longer true in general relativity).

The post-Newtonian computation of neutral modes by Cutler and Lindblom [114, 115] has shown that general relativity tends to strengthen the CFS-instability. Compared to their Newtonian counterparts, critical angular velocity ratios  $\Omega_c/\Omega_0$  (where  $\Omega_0 = (3M_0/4R_0^3)^{1/2}$  and  $M_0, R_0$  are the mass and radius of the nonrotating star in the sequence), are lowered by as much as 10% for stars obeying the  $N = 1$  polytropic EOS (for which the instability occurs only for  $l = m \geq 3$  modes in the post-Newtonian approximation).

In full general relativity, neutral modes have been determined for polytropic EOSs of  $N \geq 1.0$  by Stergioulas and Friedman [42, 92], using a new numerical scheme. The scheme completes the Eulerian formalism developed by Ipser and Lindblom in the Cowling approximation (where  $\delta g_{ab}$  was neglected) [124], by finding an appropriate gauge in which the time-independent perturbation equations can be solved numerically for  $\delta g_{ab}$ . Because linear perturbations have a gauge freedom, four out of ten components of  $\delta g_{ab}$  are fixed by the choice of gauge. In the Ipser and Lindblom scheme, the perturbed Euler equations are solved analytically. A complete neutral mode solution of the perturbation equations is then determined by setting the frequency in the inertial frame equal to zero and solving six perturbed field equations for  $\delta g_{ab}$  and the perturbed equation of energy conservation for a scalar function  $\delta U$ .

The six perturbed field equations in the gauge of Stergioulas and Friedman are of different types. Three are second order ODEs, two are elliptic and the other one is parabolic. Their solutions vanish at the center, at infinity and on the axis of symmetry, while they are either odd or even under reflection in the equatorial plane. The six equations, although of different type, are solved simultaneously on a two-dimensional grid, which extends to infinity by a redefinition of the radial variable. Solutions of the perturbed field equations are obtained for a set of trial functions  $\delta U_i$ . In the Newtonian limit, it was found that the real eigenfunctions can be expanded accurately in terms of these trial functions [81].

The remaining equation to be satisfied, the perturbed energy conservation equation, can be represented schematically as a linear operator  $L$  on the eigenfunction  $\delta U$ . Defining an inner product  $\langle \delta U_j | L | \delta U_i \rangle$ , for the set of trial functions, the perturbed energy conservation equation is satisfied, when

$$\det \langle \delta U_j | L | \delta U_i \rangle = 0. \quad (25)$$

Using this criterion, one starts with slowly rotating configurations and increases the angular velocity of the star until (25) is satisfied and a complete neutral mode solution is obtained.

The determination of neutral modes for  $N = 1.0, 1.5$  and  $2.0$  relativistic polytropes shows that relativity significantly strengthens the instability (which was already indicated in the post-Newtonian approximation). For the

$N = 1.0$  polytrope, the critical angular velocity ratio  $\Omega_c/\Omega_K$ , where  $\Omega_K$  is the angular velocity at the mass-shedding limit at same central energy density, drops by as much as 15% for the most relativistic configuration. This is a large decrease compared to the Newtonian values, which significantly moves the onset of the instability away from the mass-shedding limit and which strengthens it with respect to the damping effect of viscosity.

A surprising result, which was not detected in the post-Newtonian approximation, is that the  $l = m = 2$  bar mode is unstable for relativistic polytropes of index  $N = 1.0$ . The classical Newtonian result for the onset of the bar mode instability ( $N_{crit} < 0.808$ ) is replaced by

$$N_{crit} < 1.3, \quad (26)$$

in general relativity.

Also, in relativistic stars, the onset of the gravitational radiation driven bar mode is different from the onset of the viscosity driven bar mode. While in the Newtonian limit the two bar modes occur at the same critical rotation ratio [79], relativity strengthens the gravitational radiation instability, allowing softer configurations to become unstable and suppresses the viscosity driven instability allowing it to occur only for very stiff EOSs [131].

An independent determination of the onset of the CFS-instability in the relativistic Cowling approximation by Yoshida and Eriguchi [121] agrees qualitatively with the conclusions in [92].

Morsink, Stergioulas and Blattning [132] extend the method presented in [92] to a wide range of realistic equations of state (which usually have a stiff high density region, corresponding to polytropes of index  $N = 0.5 - 0.7$ ) and find that the  $l = m = 2$  bar mode becomes unstable for stars with gravitational mass as low as  $1.0 - 1.2M_\odot$ . For  $1.4M_\odot$  neutron stars, the mode becomes unstable at 80% – 95% of the maximum allowed rotation rate. For a wide range of equations of state, the  $l = m = 2$   $f$ -mode becomes unstable at a ratio of rotational to gravitational energies  $T/W \sim 0.08$  for  $1.4M_\odot$  stars and  $T/W \sim 0.06$  for maximum mass stars. This is to be contrasted with the Newtonian value of  $T/W \sim 0.14$ . The empirical formula

$$(T/W)_2 = 0.115 - 0.048 \frac{M}{M_{\max}^{\text{sph}}}, \quad (27)$$

where  $M_{\max}^{\text{sph}}$  is the maximum mass for a spherical star allowed by a given equation of state, gives the critical value of  $T/W$  for the bar  $f$ -mode instability, with an accuracy of 4% – 6%, independent of the equation of state.

Conservation of angular momentum and the inferred initial period (assuming magnetic braking) of 6 – 9ms for the X-ray pulsar in the supernova remnant N157B [133], suggests that a fraction of neutron stars may be born with very large rotational energies. The  $f$ -mode bar CFS-instability thus appears as a promising source for the planned gravitational wave detectors [134]. It could also play a major role in the rotational evolution, through the

emission of gravitational waves, of merged binary neutron stars, if their post-merger angular momentum exceeds the maximum allowed to form a Kerr black hole [135].

### 3.5.3 CFS-Instability of Axial Modes

In nonrotating stars, axial fluid modes are degenerate at zero-frequency but in rotating stars they have nonzero frequency and are called  $r$ -modes in the Newtonian limit [136, 137]. To  $O(\Omega)$ , their frequency in the inertial frame is

$$\omega_i = -m\Omega \left(1 - \frac{2}{l(l+1)}\right), \quad (28)$$

and modes with different radial eigenfunctions can be computed at order  $\Omega^2$  [112, 138]. According to (28),  $r$ -modes with  $m > 0$  are prograde ( $\omega_i < 0$ ) with respect to a distant observer but retrograde ( $\omega_r = \omega_i + m\Omega > 0$ ) in the comoving frame for all values of the angular velocity. Thus,  $r$ -modes in relativistic stars are generically unstable to the emission of gravitational waves via the CFS-instability, which was first discovered by Andersson [139], in the case of slowly rotating, relativistic stars. This result is confirmed analytically by Friedman and Morsink [140], who show that the canonical energy of the modes is negative.

Two independent computations in the Newtonian Cowling approximation, by Andersson, Kokkotas and Schutz [127] and Lindblom, Owen and Morsink [141] show that viscosity is not able to damp the  $r$ -mode instability in rotating stars. In a temperature window of  $10^5 \text{ K} < T < 10^{10} \text{ K}$ , the growth time of the  $l = m = 2$  mode becomes shorter than the shear or bulk viscosity damping time at a critical rotation rate that is roughly one tenth the maximum allowed angular velocity of uniformly rotating stars. The gravitational radiation is dominated by the current quadrupole term. These results suggest that a rapidly rotating proto-neutron star will spin down to Crab-like rotation rates within one year of its birth, because of the  $r$ -mode instability. The current uncertainties in the viscosity and superfluid mutual friction damping times make this scenario also consistent with somewhat higher initial spins, like the suggested initial spin of 6 – 9ms of the X-ray pulsar in the supernova remnant N157B [133]. Millisecond pulsars with periods less than  $\sim 5\text{ms}$  can then only form after the accretion-induced spin-up of old pulsars and not in the accretion-induced collapse of a white dwarf.

The precise limit on the angular velocity of newly-born neutron stars will depend on several factors, such as the strength of the bulk viscosity, the cooling process, the superfluid mutual friction etc. In the uniform density approximation, the  $r$ -mode instability can be studied analytically to  $O(\Omega^2)$  in the angular velocity of the star and the resulting expressions for the timescales, given in Kokkotas and Stergioulas [142], can be used to study the effect of such factors on the instability. In [142] it is also shown that the minimum critical angular velocity for the

onset of the  $r$ -mode instability is rather insensitive to the choice of equation of state.

A first study on the issue of detectability of gravitational waves from the  $r$ -mode instability, is presented in [144] (see section 3.5.6), while Andersson, Kokkotas and Stergioulas [143] study the relevance of the  $r$ -mode instability in limiting the spin of recycled millisecond pulsars.

### 3.5.4 Effect of Viscosity on CFS-Instability

In the previous sections, we have discussed the growth of the CFS-instability driven by gravitational radiation in an otherwise nondissipative star. The effect of neutron star matter viscosity on the dynamical evolution of non-axisymmetric perturbations can be considered separately, when the timescale of the viscosity is much longer than the oscillation timescale. If  $\tau_{GR}$  is the computed growth rate of the instability in the absence of viscosity and  $\tau_s$ ,  $\tau_b$  are the timescales of shear and bulk viscosity, then the total timescale of the perturbation is

$$\frac{1}{\tau} = \frac{1}{\tau_{GR}} + \frac{1}{\tau_s} + \frac{1}{\tau_b}. \quad (29)$$

Since  $\tau_{GR} < 0$  and  $\tau_b, \tau_s > 0$ , a mode will grow only if  $\tau_{GR}$  is shorter than the viscous timescales, so that  $1/\tau < 0$ .

The shear and bulk viscosity are sensitive to several factors and we give here a summary of what is known to date from Newtonian and post-Newtonian computations:

- Shear viscosity

In normal neutron star matter, shear viscosity is dominated by neutron-neutron scattering with a temperature dependence of  $T^{-2}$  [145] and computations in the Newtonian limit and post-Newtonian approximation show that the CFS-instability is suppressed for  $T < 10^6 \text{ K} - 10^7 \text{ K}$  [82, 83, 125, 115].

If neutrons become a superfluid below a transition temperature  $T_s$ , then mutual friction, which is caused by the scattering of electrons off the cores of neutron vortices can completely suppress the instability for  $T < T_s$ . The superfluid transition temperature depends on the theoretical model for superfluidity and lies in the range  $10^8 \text{ K} - 6 \times 10^9 \text{ K}$  [146].

- Bulk Viscosity

In a pulsating fluid that undergoes compression and expansion, the weak interaction requires a relatively long time to re-establish equilibrium. This creates a phase lag between density and pressure perturbations, which results in a large bulk viscosity [147]. The bulk viscosity due to this effect can suppress the CFS-instability only for temperatures for which matter has become transparent to neutrinos. [134], [148]. It has been proposed that for  $T > 5 \times 10^9 \text{ K}$ , matter will be opaque to neutrinos and the neutrino phase space could be blocked ([134] see also [148]). In this case, bulk viscosity will be too weak to suppress the instability, but a more detailed study is needed.

In the neutrino transparent regime, the effect of bulk viscosity on the instability depends crucially on the proton fraction  $x_p$ . If  $x_p$  is lower than a critical value ( $\sim \frac{1}{9}$ ), only modified URCA processes are allowed and bulk viscosity limits, but does not suppress completely, the instability [82, 83, 125]. For most modern EOSs, however, the proton fraction is larger than  $\sim \frac{1}{9}$  at sufficiently high densities [151], allowing direct URCA processes to take place. In this case, depending on the EOS and the central density of the star, the bulk viscosity could almost completely suppress the CFS-instability in the neutrino transparent regime [152] (but it will probably still not affect it for temperatures  $T > 5 \times 10^9$  K).

In conclusion, the available Newtonian computations indicate that the CFS-instability in  $f$ -modes is effective in nascent neutron stars for temperatures between  $10^9$ K and  $10^{10}$ K and possibly also above  $10^{10}$ K if the star is opaque to neutrinos and the bulk viscosity is weak. If direct URCA reactions do participate in the cooling process, it appears that the instability can grow only for temperatures for which the star is opaque to neutrinos. Since the neutral mode computations in fully relativistic stars show that relativity strengthens the instability, the above conclusion should also hold in relativistic stars.

The uncertainties regarding the effect of viscosity on the CFS-instability in realistic neutron stars will be greatly reduced by the construction of mode eigenfunctions for fully relativistic, rotating stars.

### 3.5.5 Viscosity-Driven Instability

A different type of nonaxisymmetric instability in rotating stars is that driven by viscosity, which breaks the circulation of the fluid [153, 129]. The instability is suppressed by gravitational radiation, so it can act only in cold neutron stars that become rapidly rotating by accretion-induced spin-up. The instability sets in when the frequency of an  $l = -m$  mode goes through zero in the rotating frame. In contrast to the CFS-instability, the viscosity-driven instability is not generic in rotating stars. The  $m = 2$  mode becomes unstable at a high rotation rate for very stiff stars and higher  $m$ -modes become unstable at larger rotation rates.

In Newtonian polytropes, the instability occurs only for stiff polytropes of index  $N < 0.808$  [129, 130]. For relativistic models, the situation for the instability becomes worse, since relativistic effects tend to suppress the viscosity instability (while they strengthen the CFS-instability). According to recent results by Bonazzola et al. [131], for the most relativistic stars, the viscosity driven bar mode can become unstable only if  $N < 0.55$ . For  $1.4M_\odot$  stars, the instability is present for  $N < 0.67$ .

These results are based on an approximate computation of the instability in which one perturbs an axisymmetric and stationary configuration and studies its evolution by constructing a series of triaxial quasi-equilibrium configurations. During the evolution only the dominant

nonaxisymmetric terms are taken into account.

The method presented in [131] is an improvement (taking into account nonaxisymmetric terms of higher order) of an earlier method by the same authors [148]. Although the method is approximate, its results indicate that the viscosity-driven instability is likely to be absent in most relativistic stars, unless the EOS turns out to be unexpectedly stiff.

An investigation of the viscosity-driven bar mode instability, using incompressible, uniformly rotating triaxial ellipsoids in the post-Newtonian approximation, by Shapiro and Zane [149], also finds that the relativistic effects weaken the instability.

### 3.5.6 Gravitational Radiation from CFS-Instability

The CFS-instability can limit the maximum angular velocity of nascent neutron stars, but it is also a mechanism for the generation of gravitational waves that could be strong enough to be detected by the planned gravitational wave detectors.

Lai and Shapiro [134] have studied the development of the  $f$ -mode instability using Newtonian ellipsoidal rotating models [154, 155]. They consider the case where a rapidly rotating neutron star is created in a core collapse. After a brief dynamical phase, the protoneutron star becomes axisymmetric but secularly unstable. The instability deforms the star into a nonaxisymmetric configuration via the  $l = 2$  bar mode. Since the star loses angular momentum via the emission of gravitational waves, it spins down until it becomes secularly stable.

The frequency of the waves sweeps downward from a few hundred Hz to zero, passing through LIGO's ideal sensitivity band. A rough estimate of the wave amplitude shows that, at  $\sim 100$ Hz, the gravitational waves from the CFS-instability could be detected out to the distance of 140Mpc by the advanced LIGO detector. This result is very promising, especially since for relativistic stars the instability will be stronger than the present Newtonian estimate.

The recently discovered CFS-instability in  $r$ -modes, is also an important source of gravitational waves. Owen et al. [144] model the development of the instability and the evolution of the neutron star during its spin-down phase. The evolution suggests that a neutron star formed in the Virgo cluster could be detected by the advanced LIGO and VIRGO gravitational wave detectors, with an amplitude signal-to-noise ratio that could be as large as about 8, if near-optimal data analysis techniques are developed. Assuming a substantial fraction of neutron stars are born with spin frequencies near their maximum values, the stochastic background of gravitational waves produced by the  $r$ -mode radiation from neutron star formation throughout the universe is shown to have an energy density of about  $10^{-9}$  of the cosmological closure density, in the range 20 Hz to 1 kHz. This radiation is potentially



detectable by the advanced LIGO as well.

In newly born stars or in the post-merger objects in binary neutron star mergers, rotating close to the Kepler limit, both the  $f$  and  $r$  modes will be unstable. Relativistic computations of growth times in rapidly rotating stars or even nonlinear evolutions, are needed to determine which mode will be strongest.

- **Going further.** The possible ways for neutron stars to emit gravitational waves and on their detectability are reviewed by Bonazzola, Gourgoulhon, Flanagan, Thorne and Schutz in [156, 157, 158, 159, 160, 161].

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